Forbidding rank-preserving copies of a poset

Dániel Gerbner[a,*], Abhishek Methuku[b], Dániel T. Nagy[a,†], Balázs Patkós[a,‡], Máté Vizer[a,§]

[a] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
P.O.B. 127, Budapest H-1364, Hungary.

[b] Central European University, Department of Mathematics
Budapest, H-1051, Nádor utca 9.

{gerbner,nagydani,patkos}@renyi.hu, {abhishekmethuku,vizermate}@gmail.com

September 28, 2018

Abstract

The maximum size, $La(n,P)$, of a family of subsets of $[n] = \{1,2,...,n\}$ without containing a copy of P as a subposet, has been intensively studied.

Let P be a graded poset. We say that a family $\mathcal{F}$ of subsets of $[n] = \{1,2,...,n\}$ contains a *rank-preserving* copy of P if it contains a copy of P such that elements of P having the same rank are mapped to sets of same size in $\mathcal{F}$. The largest size of a family of subsets of $[n] = \{1,2,...,n\}$ without containing a rank-preserving copy of P as a subposet is denoted by $La_{rp}(n,P)$. Clearly, $La(n,P) \le La_{rp}(n,P)$ holds.

In this paper we prove asymptotically optimal upper bounds on $La_{rp}(n,P)$ for tree posets of height 2 and monotone tree posets of height 3, strengthening a result of Bukh in these cases. We also obtain the exact value of $La_{rp}(n,\{Y_{h,s},Y'_{h,s}\})$ and $La(n,\{Y_{h,s},Y'_{h,s}\})$, where $Y_{h,s}$ denotes the poset on $h+s$ elements $x_1,\dots,x_h,y_1,\dots,y_s$ with $x_1 < \cdots < x_h < y_1,\dots,y_s$ and $Y'_{h,s}$ denotes the dual poset of $Y_{h,s}$.

*Research supported by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and the National Research, Development and Innovation Office – NKFIH under the grant K 116769.

†Research supported by the ÚNKP-17-3 New National Excellence Program of the Ministry of Human Capacities and by National Research, Development and Innovation Office - NKFIH under the grant K 116769.

‡Research supported by the National Research, Development and Innovation Office – NKFIH under the grants SNN 116095 and K 116769.

§Research supported by the National Research, Development and Innovation Office – NKFIH under the grant SNN 116095.

1 Introduction

In extremal set theory, many of the problems considered can be phrased in the following way: what is the size of the largest family of sets that satisfy a certain property. The very first such result is due to Sperner [15] which states that if $\mathcal{F}$ is a family of subsets of $[n] = \{1, 2 \dots, n\}$ (we write $\mathcal{F} \subseteq 2^{[n]}$ to denote this fact) such that no pair $F, F' \in \mathcal{F}$ of sets are in inclusion $F \subsetneq F'$, then $\mathcal{F}$ can contain at most $\binom{n}{\lfloor n/2 \rfloor}$ sets. This is sharp as shown by $\binom{[n]}{\lfloor n/2 \rfloor}$ (the family of all k-element subsets of a set X is denoted by $\binom{X}{k}$ and is called the k^{th} *layer* of X). If P is a poset, we denote by $\leq_P$ the partial order acting on the elements of P. Generalizing Sperner's result, Katona and Tarján [8] introduced the problem of determining the size of the largest family $\mathcal{F} \subseteq 2^{[n]}$ that does not contain sets satisfying some inclusion patterns. Formally, if P is a finite poset, then a subfamily $\mathcal{G} \subseteq \mathcal{F}$ is

- a *(weak) copy* of P if there exists a bijection $\phi : P \to \mathcal{G}$ such that we have $\phi(x) \subseteq \phi(y)$ whenever $x \leq_P y$ holds,
- a *strong* or *induced copy* of P if there exists a bijection $\phi : P \to \mathcal{G}$ such that we have $\phi(x) \subseteq \phi(y)$ if and only if $x \leq_P y$ holds.

A family is said to be *P-free* if it does not contain any (weak) copy of P and *induced P-free* if it does not contain any induced copy of P. Katona and Tarján started the investigation of determining

$$La(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]},\ \mathcal{F} \text{ is } P\text{-free}\}$$

and

$$La^*(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]},\ \mathcal{F} \text{ is induced } P\text{-free}\}.$$

The above quantities have been determined precisely or asymptotically for many classes of posets (see [6] for a nice survey), but the question has not been settled in general. Recently, Methuku and Pálvölgyi [11] showed that for any poset P, there exists a constant C_P such that $La(n, P) \leq La^*(n, P) \leq C_P\binom{n}{\lfloor n/2 \rfloor}$ holds (the inequality $La(n, P) \leq |P|\binom{n}{\lfloor n/2 \rfloor}$ follows trivially from a result of Erdős [4]). However, it is still unknown whether the limits $\pi(P) = \lim_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}$ and $\pi^*(P) = \lim_{n\to\infty} \frac{La^*(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}$ exist. In all known cases, the asymptotics of $La(n, P)$ and $La^*(n, P)$ were given by "taking as many middle layers as possible without creating an (induced) copy of P". Therefore researchers of the area believe the following conjecture that appeared first in print in [7].

Conjecture 1.1. *(i) For any poset P let $e(P)$ denote the largest integer k such that for any j and n the family $\cup_{i=1}^k \binom{[n]}{j+i}$ is P-free. Then $\pi(P)$ exists and is equal to $e(P)$.*

(ii) For any poset P let $e^*(P)$ denote the largest integer k such that for any j and n the family $\cup_{i=1}^k \binom{[n]}{j+i}$ is induced P-free. Then $\pi^*(P)$ exists and is equal to $e^*(P)$.

Let P be a graded poset with rank function ρ. Given a family $\mathcal{F}$, a subfamily $\mathcal{G} \subseteq \mathcal{F}$ is a rank-preserving copy of P if $\mathcal{G}$ is a (weak) copy of P such that elements having the same rank in P are mapped to sets of same size in $\mathcal{G}$. More formally, $\mathcal{G} \subseteq \mathcal{F}$ is a rank-preserving copy of P if there is a bijection $\phi : P \to \mathcal{G}$ such that $|\phi(x)| = |\phi(y)|$ whenever $\rho(x) = \rho(y)$ and we have $\phi(x) \subseteq \phi(y)$ whenever $x \leq_P y$ holds. A family $\mathcal{F}$ is rank-preserving P-free if it does not contain a rank-preserving copy of P. In this paper, we study the function

$$La_{rp}(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]},\ \mathcal{F} \text{ is rank-preserving } P\text{-free}\}.$$

In fact, our problem is a natural special case of the following general problem introduced by Nagy [13]. Let $c : P \to [k]$ be a coloring of the poset P such that for any $x \in [k]$ the pre-image $c^{-1}(x)$ is an antichain. A subfamily $\mathcal{G} \subseteq \mathcal{F}$ is called a *c-colored copy* of P in $\mathcal{F}$ if $\mathcal{G}$ is a (weak) copy of P and sets corresponding to elements of P of the same color have the same size. Nagy investigated the size of the largest family $\mathcal{F} \subseteq 2^{[n]}$ which does not contain a c-colored copy of P, for several posets P and colorings c. Note that when c is the rank function of P, then this is equal to $La_{rp}(n, P)$. Nagy also showed that there is a constant C_P such that $La_{rp}(n, P) \leq C_P\binom{n}{\lfloor n/2 \rfloor}$.

A complete multi-level poset is a poset in which every element of a level is related to every element of another level. Note that any rank-preserving copy of a complete multi-level poset P is also an induced copy of P. In fact, in [14], Patkós determined the asymptotics of $La^*(n, P)$, for some complete multi-level posets P by finding a rank preserving copy of P.

By definition, for every graded poset P we have $La(n, P) \leq La_{rp}(n, P)$. Boehnlein and Jiang [1] gave a family of posets P showing that the difference between $La^*(n, P)$ and $La(n, P)$ can be arbitrarily large. Since their posets embed into a complete multi-level poset of height 3 in a rank-preserving manner, the above mentioned result of Patkós implies that for the same family of posets, $La_{rp}(n, P)$ can be arbitrarily smaller than $La^*(n, P)$. However, it would be interesting to determine if the opposite phenomenon can occur.

1.1 Our results

Asymptotic results

For a poset P its *Hasse diagram*, denoted by $H(P)$, is a graph whose vertices are elements of P, and xy is an edge if $x < y$ and there is no other element z of P with $x < z < y$. We call a poset, *tree poset* if $H(P)$ is a tree. A tree poset is called *monotone increasing* if it has a unique minimal element and it is called *monotone decreasing* if it has a unique maximal element. A tree poset is *monotone* if it is either monotone increasing or decreasing.

A remarkable result concerning Conjecture 1.1 is that of Bukh [2], who verified Conjecture 1.1 (i) for tree posets. In the following results we strengthen his result in two cases.

Theorem 1.2. *Let T be any tree poset of height 2. Then we have*

$$La_{rp}(n, T) = \left(1 + O_T\left(\sqrt{\frac{\log n}{n}}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Theorem 1.3. *Let T be any monotone tree poset of height 3. Then we have*

$$La_{rp}(n,T) = \left(2 + O_T\left(\sqrt{\frac{\log n}{n}}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The lower bounds in Theorem 1.2 and Theorem 1.3 follow simply by taking one and two middle layers of the Boolean lattice of order n, respectively.

An exact result

The *dual* of a poset P is the poset P' on the same set with the partial order relation of P replaced by its inverse, i.e., $x \le y$ holds in P if and only if $y \le x$ holds in P'. Let $Y_{h,s}$ denote the poset on $h+s$ elements $x_1, \dots, x_h, y_1, \dots, y_s$ with $x_1 < \dots < x_h < y_1, \dots, y_s$ and let $Y'_{h,s}$ denote the dual of $Y_{h,s}$. Let $\Sigma(n,h)$ for the number of elements on the h middle layers of the Boolean lattice of order n, so $\Sigma(n,h) = \sum_{i=1}^{h} \binom{n}{\lfloor \frac{n-h}{2} \rfloor + i}$.

Investigation on $La(n, Y_{h,s})$ was started by Thanh in [16], where asymptotic results were obtained. Thanh also gave a construction showing that $La(n, Y_{h,s}) > \Sigma(n,h)$, from which it easily follows that $La(n, Y'_{h,s}) > \Sigma(n,h)$ as well. Interestingly, De Bonis and Katona [3] showed that if both $Y_{2,2}$ and $Y'_{2,2}$ are forbidden, then an exact result can be obtained: $La(n, \{Y_{2,2}, Y'_{2,2}\}) = \Sigma(n,2)$. Later this was extended by Methuku and Tompkins [12], who proved $La(n, \{Y_{k,2}, Y'_{k,2}\}) = \Sigma(n,k)$, and $La^*(n, \{Y_{2,2}, Y'_{2,2}\}) = \Sigma(n,2)$. Very recently, Martin, Methuku, Uzzell and Walker [10] and independently, Tompkins and Wang [17] showed that $La^*(n, \{Y_{k,2}, Y'_{k,2}\}) = \Sigma(n,k)$. We prove the following theorem which extends all of these previous results and proves a conjecture of [10].

Theorem 1.4. *For any pair $s, h \ge 2$ of positive integers, there exists $n_0 = n_0(h,s)$ such that for any $n \ge n_0$ we have*

$$La_{rp}(n, \{Y_{h,s}, Y'_{h,s}\}) = \Sigma(n,h).$$

The lower bound trivially follows by taking h middle layers of the Boolean lattice of order n. (Note that adding any extra set creates a rank-preserving copy of either $Y_{h,s}$ or $Y'_{h,s}$.) Moreover, any rank-preserving copy of $Y_{h,s}$ (respectively $Y'_{h,s}$) is also an induced copy of $Y_{h,s}$ (respectively $Y'_{h,s}$). Therefore, Theorem 1.4 implies that $La^*(n, \{Y_{h,s}, Y'_{h,s}\}) = La(n, \{Y_{h,s}, Y'_{h,s}\}) = \Sigma(n,h)$.

Remark. One wonders if the condition $h \ge 2$ is necessary in Theorem 1.4. Katona and Tarján [8] proved that $La(n, \{Y_{1,2}, Y'_{1,2}\}) = \binom{n}{n/2}$ if n is even and $La(n, \{Y_{1,2}, Y'_{1,2}\}) = 2\binom{n-1}{(n-1)/2} > \binom{n}{n/2}$ if n is odd. The following construction shows that no matter how little we weaken the condition of being $\{Y_{1,2}, Y'_{1,2}\}$-free, there are families strictly larger than $\binom{n}{n/2}$ even in the case n is even. Let us define

$$\mathcal{F}_{2,3} = \left\{ F \in \binom{[n]}{n/2+1} : n-1, n \in F \right\} \cup \left\{ F \in \binom{[n]}{n/2} : |F \cap \{n-1, n\}| \le 1 \right\}.$$

Observe that $\mathcal{F}_{2,3}$ is $\{Y_{1,2}, Y'_{1,3}\}$-free and its size is $\binom{n-2}{n/2+1} + (\binom{n}{n/2} - \binom{n-2}{n/2-2}) > \binom{n}{n/2}$.

2 Proofs

Using Chernoff's inequality, it is easy to show (see for example [7]) that the number of sets $F \subset [n]$ of size more than $n/2 + 2\sqrt{n \log n}$ or smaller than $n/2 - 2\sqrt{n \log n}$ is at most

$$O\left(\frac{1}{n^{3/2}}\binom{n}{n/2}\right). \tag{1}$$

Thus in order to prove Theorem 1.2 and Theorem 1.3, we can assume the family only contains sets of size more than $n/2 - 2\sqrt{n \log n}$ and smaller than $n/2 + 2\sqrt{n \log n}$.

2.1 Proof of Theorem 1.2: Trees of height two

The proof of Theorem 1.2 follows the lines of a reasoning of Bukh's [2]. The new idea is that we count the number of related pairs between two fixed levels as detailed in the proof below.

Let $\mathcal{F}$ be a T-free family of subsets of $[n]$ and let the number of elements in T be t. Using (1), we can assume $\mathcal{F}$ only contains sets of sizes in the range $[n/2 - 2\sqrt{n \log n}, n/2 + 2\sqrt{n \log n}]$. A pair of sets $A, B \in \mathcal{F}$ with $A \subset B$ is called a 2-chain in $\mathcal{F}$. It is known by a result of Kleitman [9] that the number of 2-chains in $\mathcal{F}$ is at least

$$\left(|\mathcal{F}| - \binom{n}{\lfloor \frac{n}{2} \rfloor}\right)\frac{n}{2}. \tag{2}$$

For any $n/2 - 2\sqrt{n \log n} \leq i \leq n/2 + 2\sqrt{n \log n}$, let $\mathcal{F}_i := \mathcal{F} \cap \binom{[n]}{i}$.

Claim 2.1. For any $i < j$, the number of 2-chains $A \subset B$ with $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ is at most $(t-2)(|\mathcal{F}_i| + |\mathcal{F}_j|)$.

Proof. Suppose otherwise, and construct an auxiliary graph G whose vertices are elements of $\mathcal{F}_i$ and $\mathcal{F}_j$, and two vertices form an edge of G if the corresponding elements form a 2-chain. This implies that G contains more than $(t-2)(|\mathcal{F}_i| + |\mathcal{F}_j|)$ edges, so it has average degree more than $2(t-2)$. One can easily find a subgraph G' of G with minimum degree at least $t-1$, into which we can greedily embed any tree with t vertices. So in particular, we can find T in G' which corresponds to a rank-preserving copy of T into $\mathcal{F}$, a contradiction. $\square$

Claim 2.1 implies that the total number of 2-chains in $\mathcal{F}$ is at most

$$\sum_{n/2 - 2\sqrt{n \log n} \leq i < j \leq n/2 + 2\sqrt{n \log n}} (t-2)(|\mathcal{F}_i| + |\mathcal{F}_j|) = (t-2)(4\sqrt{n \log n})\,|\mathcal{F}|\,.$$

Combining this with (2), and simplifying we get

$$|\mathcal{F}|\left(1 - 8(t-2)\sqrt{\frac{\log n}{n}}\right) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Rearranging, we get

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + O_T\left(\sqrt{\frac{\log n}{n}}\right)\right)$$

as desired.

$\square$

2.2 Proof of Theorem 1.3: Monotone trees of height three

First note that it is enough to prove the statement for $T = T_{r,3}$ the monotone increasing tree poset of height tree where all elements, except its leaves (i.e., its elements on the top level) have degree r. Let $\mathcal{F} \subseteq 2^{[n]}$ be a family of sets which does not contain any rank-preserving copies of $T_{r,3}$. Using (1) we can assume that for any set $F \in \mathcal{F}$ we have $|F - n/2| \leq 2\sqrt{n \log n}$.

We will prove that for such a family,

$$\sum_{F \in \mathcal{F}} |F|!(n - |F|)! \leq (2 + O_r(1/n))n! \tag{3}$$

holds. This is enough as dividing by $n!$ yields

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq (2 + O_r(1/n))$$

and hence the statement of the theorem will follow.

Observe that $\sum_{F \in \mathcal{F}} |F|!(n - |F|)!$ is the number of pairs $(F, \mathcal{C})$ where $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C}$ is a maximal chain in $[n]$. We will use the chain partitioning method introduced in [5]. For any $G \in \mathcal{F}$ we define $\mathbf{C}_G$ to be the set of maximal chains $\mathcal{C}$ in $[n]$ such that the smallest set of $\mathcal{C} \cap \mathcal{F}$ is G.

To prove (3) it is enough to show that for any fixed $G \in \mathcal{F}$ the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$, $\mathcal{C} \in \mathbf{C}_G$ is at most $(2 + O_r(1/n))|\mathbf{C}_G|$. We count the number of these pairs $(F, \mathcal{C})$ in three parts.

Firstly, the number of pairs where either $F = G$ or F is the second smallest element of $\mathcal{F} \cap \mathcal{C}$ is at most $2|\mathbf{C}_G|$ (there might be chains in $\mathbf{C}_G$ with $\mathcal{C} \cap \mathcal{F} = \{G\}$).

Let us consider the following sub-partition of $\mathbf{C}_G$. For any $G \subsetneq G' \in \mathcal{F}$ let $\mathbf{C}_{G,G'}$ denote the set of maximal chains $\mathcal{C}$ such that G and G' are the smallest and second smallest sets in $\mathcal{F} \cap \mathcal{C}$, respectively. Observe that $|\mathbf{C}_{G,G'}| = m_G \cdot m_{G,G'} \cdot (n - |G'|)!$, where m_G is the number of chains from $\emptyset$ to G that do not contain any other sets from $\mathcal{F}$ and $m_{G,G'}$ is the number of chains from G to G' that do not contain any other sets from $\mathcal{F}$.

Secondly, let us now count the pairs $(F, \mathcal{C})$ such that $F \in \mathcal{F} \cap \mathcal{C}$, $\mathcal{C} \in \mathbf{C}_{G,G'}$ and there are less than r^2 sets $F' \in \mathcal{F}$ with $|F'| = |F|$, $G' \subsetneq F'$. To this end, let us fix G' and count such pairs $(F, \mathcal{C})$. All sets in $\mathcal{F}$ have size at most $n/2 + 2\sqrt{n \log n}$ and at least $n/2 - 2\sqrt{n \log n}$, so

$|G'| \geq n/2 - 2\sqrt{n \log n}$. For a set $F \supsetneq G'$ the number of chains in $\mathbf{C}_{G,G'}$ that contain F is $m_G m_{G,G'} \cdot (|F| - |G'|)!(n - |F|)!$, thus we obtain that the number of such pairs is at most

$$\sum_{i=1}^{4\sqrt{n \log n}} r^2 m_G m_{G,G'} \cdot i!(n-|G'|-i)! \leq 2r^2 m_G m_{G,G'}(n-|G'|-1)! = \frac{2r^2}{n-|G'|}|\mathbf{C}_{G,G'}| \leq \frac{5r^2}{n}|\mathbf{C}_{G,G'}|.$$

Summing this for all G' we obtain that the total number of such pairs $(F, \mathcal{C})$ of this second type is at most $\frac{5r^2}{n}|\mathbf{C}_G|$.

Finally, let us count the pairs $(F, \mathcal{C})$ with $F \in \mathcal{C} \cap \mathcal{F}$, $\mathcal{C} \in \mathbf{C}_{G,G'}$ and there are at least r^2 many sets $F' \in \mathcal{F}$ with $G' \subsetneq F'$, $|F'| = |F|$. To this end we group some of the $\mathbf{C}_{G,G'}$'s together. Let

$$\mathbf{C}_{G,k} := \cup_{G':|G'|=k} \mathbf{C}_{G,G'}, \quad \mathcal{F}_{G,k} := \{G' \in \mathcal{F} : G \subseteq G', |G'| = k\}$$

and let us introduce the function $f_{G,k} : \mathcal{F}_{G,k} \to [n]$ by

$$f_{G,k}(G') := \{j : \exists F_1, F_2, \ldots, F_{r^2}, \text{ such that } G' \subseteq F_i, |F_i| = j \text{ for all } i = 1, 2, \ldots, r^2\}.$$

Observe that for any distinct $G'_1, G'_2, \ldots, G'_r \in \mathcal{F}_{G,k}$ we have $\cap_{i=1}^k f_{G,k}(G'_i) = \emptyset$. Indeed, if $j \in \cap_{i=1}^k f_{G,k}(G'_i) = \emptyset$, then one could extend $G, G'_1, G'_2, \ldots, G'_r$ to a rank-preserving copy of $T_{r,3}$ such that all sets corresponding to leaves of $T_{r,3}$ are of size j.

Note that by the assumption on the set sizes of $\mathcal{F}$, the function $f_{G,k}$ maps to $[n/2 - 2\sqrt{n \log n}, n/2 + 2\sqrt{n \log n}]$, so its range has size at most $4\sqrt{n \log n}$. As every maximal chain contains exactly one set of size j (not necessarily contained in $\mathcal{F}$), we obtain that the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$, $\mathcal{C} \in \mathbf{C}_{G,k}$ is at most

$$m_G \cdot 4\sqrt{n \log n}(r-1)(k-|G|)!(n-k)!. \tag{4}$$

Indeed, if the size j of F is fixed, then j belongs to $f_{G,k}(G')$ for at most $r-1$ sets $G' \in \mathcal{F}_{G,k}$, so for this particular j the number of pairs is at most $m_G \cdot (r-1)(k-|G|)!(n-k)!$.

Summing up (4) for all $k > |G|$ we obtain that the number of pairs $(F, \mathcal{C})$ of this third type is at most

$$\begin{aligned}\sum_{k=|G|+1}^{n/2+2\sqrt{n \log n}} m_G \cdot 4\sqrt{n \log n}(r-1)(k-|G|)!(n-k)! &\leq \frac{8(r-1)\sqrt{n \log n}}{n-|G|} m_G(n-|G|)! \\ &\leq \frac{17(r-1)\sqrt{n \log n}}{n}|\mathbf{C}_G|.\end{aligned}$$

Adding up the estimates on the number of pairs $(F, \mathcal{C})$ of these 3 types, completes the proof. □

2.3 Proof of Theorem 1.4: $\{Y_{h,s}, Y'_{h,s}\}$-free families

Let $\mathcal{F} \subset 2^{[n]}$ be a family not containing a rank-preserving copy of $Y_{h,s}$ or $Y'_{h,s}$. First, we will introduce a weight function. For every $F \in \mathcal{F}$, let $w(F) = \binom{n}{|F|}$. For a maximal chain $\mathcal{C}$, let $w(\mathcal{C}) = \sum_{F \in \mathcal{C} \cap \mathcal{F}} w(F)$ denote the weight of $\mathcal{C}$. Let $\mathbf{C}_n$ denote the set of maximal chains in $[n]$. Then

$$\frac{1}{n!} \sum_{\mathcal{C} \in \mathbf{C}_n} w(\mathcal{C}) = \frac{1}{n!} \sum_{\mathcal{C} \in \mathbf{C}_n} \sum_{F \in \mathcal{C} \cap \mathcal{F}} w(F) = \frac{1}{n!} \sum_{F \in \mathcal{F}} |F|!(n - |F|)! w(F) = |\mathcal{F}|.$$

This means that the average of the weight of the full chains equals the size of $\mathcal{F}$. Therefore it is enough to find an upper bound on this average. We will partition $\mathbf{C}_n$ into some parts and show that the average weight of the chains is at most $\Sigma(n,h)$ in each of the parts. Therefore this average is also at most $\Sigma(n,h)$, when calculated over all maximal chains, which gives us $|\mathcal{F}| \le \Sigma(n,h)$.

Let $\mathcal{G} = \{F \in \mathcal{F} \mid \exists P, Q \in \mathcal{F} \backslash \{F\},\ P \subset F \subset Q\}$. Let $A_1 \subset A_2 \subset \dots \subset A_{h-1}$ be $h-1$ different sets of $\mathcal{G}$. Then we define $\mathbf{C}(A_1, A_2, \dots, A_{h-1})$ as the set of those chains that contain all of $A_1, A_2, \dots A_{h-1}$ and these are the $h-1$ smallest elements of $\mathcal{G}$ in them. We also define $\mathbf{C}_-$ as the set of those chains that contain at most $h-2$ elements of $\mathcal{G}$. Then the sets of the form $\mathbf{C}(A_1, A_2, \dots A_{h-1})$ together with $\mathbf{C}_-$ are pairwise disjoint and their union is $\mathbf{C}_n$.

Now we will show the average weight within each of these sets of chains is at most $\Sigma(n,h)$. This is easy to see for $\mathbf{C}_-$. If $\mathcal{C} \in \mathbf{C}_-$, then $|\mathcal{C} \cap \mathcal{F}| \le h$, since every element of $\mathcal{F} \cap \mathcal{C}$ except for the smallest and the greatest must be in $\mathcal{G}$. Therefore $c(W) \le \Sigma(n,h)$ for every $\mathcal{C} \in \mathbf{C}_-$, which trivially implies

$$\sum_{\mathcal{C} \in \mathbf{C}_-} w(\mathcal{C}) \le |\mathbf{C}_-| \Sigma(n,h).$$

Now consider some sets $A_1 \subset A_2 \subset \dots \subset A_{h-1}$ in $\mathcal{G}$ such that $\mathbf{C}(A_1, A_2, \dots A_{h-1})$ is non-empty. We will use the notations $\mathbf{C}(A_1, A_2, \dots, A_{h-1}) = \mathcal{Q}$, $|A_1| = \ell_1$ and $n - |A_{h-1}| = \ell_2$ for simplicity. Note that the chains in $\mathcal{Q}$ do not contain any member of $\mathcal{F}$ of size between $|A_1|$ and $|A_{h-1}|$ other than the sets $A_2, A_3 \dots A_{h-2}$. Such a set would be in $\mathcal{G}$ (since it contains A_1 and is contained in A_{h-1}), therefore its existence would contradict the minimality of $\{A_1, A_2, \dots, A_{h-1}\}$. The chains in $\mathcal{Q}$ must also avoid all subsets of A_1 that are in $\mathcal{G}$ for the same reason.

Let N_1 denote the number of chains between $\emptyset$ and A_1 that avoid the elements of $\mathcal{G}$ (except for A_1). Let N_2 denote the number of chains between A_1 and A_{h-1} that contain the sets $A_2, A_3, \dots, A_{h-2}$, but no other element of $\mathcal{F}$. Then $|\mathcal{Q}| = N_1 N_2 \ell_2!$.

Now we will investigate how much the sets of certain sizes can contribute to the sum

$$\sum_{\mathcal{C} \in \mathcal{Q}} w(\mathcal{C}). \tag{5}$$

The sets $A_1, A_2, \dots A_{h-1}$ appear in all chains of $\mathcal{Q}$, so their contribution to the sum is

$$|\mathcal{Q}|\sum_{i=1}^{h-1} w(A_i) = |\mathcal{Q}|\sum_{i=1}^{h-1}\binom{n}{|A_i|} \le |\mathcal{Q}|\Sigma(n,h-1).$$

We have already seen that there are no other sets of $\mathcal{F}$ in these chains with a size between $|A_1|$ and $|A_{h-1}|$.

If $\ell_1 < \frac{n}{2} - 2\sqrt{n\log n}$, then (by (1)) the contribution coming from the subsets of A_1 is trivially at most

$$|\mathcal{Q}|\sum_{i=0}^{\ell_1-1}\binom{n}{i} = |\mathcal{Q}|O\left(\binom{n}{n/2}\frac{1}{n^{3/2}}\right).$$

The contribution coming from supersets of A_{h-1} is similarly small if $\ell_2 < \frac{n}{2} - 2\sqrt{n\log n}$. From now on we consider the cases when $\ell_1 \ge \frac{n}{2} - 2\sqrt{n\log n}$ and $\ell_2 \ge \frac{n}{2} - 2\sqrt{n\log n}$.

There are no s supersets of A_{h-1} of equal size in $\mathcal{F}$, since these would form a rank-preserving copy of $Y_{h,s}$ together with the sets $A_1, A_2, \dots A_{h-1}$ and some set $P \in \mathcal{F}$, $P \subset A_1$. (Such a set exists, since $A_1 \in \mathcal{G}$.)

A superset of A_{h-1} of size $n-i$ appears in $|\mathcal{Q}|\binom{\ell_2}{i}^{-1}$ chains of $\mathcal{Q}$. Therefore the total contribution to the sum (5) by supersets of A_{h-1} is at most

$$\begin{aligned}|\mathcal{Q}|w([n]) + \sum_{i=1}^{\ell_2-1}|\mathcal{Q}|\binom{\ell_2}{i}^{-1}(s-1)\binom{n}{n-i} &\le |\mathcal{Q}| + |\mathcal{Q}|(s-1)\binom{n}{\lfloor\frac{n}{2}\rfloor}\sum_{i=1}^{\ell_2-1}\binom{\ell_2}{i}^{-1}\\ &= |\mathcal{Q}|\binom{n}{\lfloor\frac{n}{2}\rfloor}O_s\left(\frac{1}{n}\right).\end{aligned}$$

There are no s subsets of A_1 of equal size in $\mathcal{F}$, since these would form a rank-preserving copy of $Y'_{h,s}$ together with the sets $A_1, A_2, \dots A_{h-1}$ and some set $Q \in \mathcal{F}$, $A_{h-1} \subset Q$. (Such a set exists, since $A_{h-1} \in \mathcal{G}$.)

A subset of A_1 of size i appears in at most $\binom{\ell_1}{i}^{-1}\ell_1!N_2\ell_2!$ chains of $\mathcal{Q}$. Therefore the total contribution to the sum (5) by subsets of A_1 is at most

$$\begin{aligned}\ell_1!N_2\ell_2!w(\emptyset) + \sum_{i=1}^{\ell_1-1}\binom{\ell_1}{i}^{-1}\ell_1!N_2\ell_2!(s-1)\binom{n}{i} &\le \ell_1!N_2\ell_2! + \ell_1!N_2\ell_2!(s-1)\binom{n}{\lfloor\frac{n}{2}\rfloor}\sum_{i=1}^{\ell_1-1}\binom{\ell_1}{i}^{-1}\\ &= \ell_1!N_2\ell_2!\binom{n}{\lfloor\frac{n}{2}\rfloor}O_s\left(\frac{1}{n}\right).\end{aligned} \tag{6}$$

We will show that if n is large and $\ell_1 \ge \frac{n}{2} - 2\sqrt{n\log n}$ then most chains between $\emptyset$ and A_1 avoid the elements of $\mathcal{G}$, therefore N_1 is close to $\ell_1!$. There are at most $s-1$ sets of $\mathcal{G}$ on any level (otherwise a rank-preserving copy of $Y'_{h,s}$ would be formed), and $\emptyset \notin \mathcal{G}$. There are $\ell_1!\binom{\ell_1}{i}^{-1}$ chains between $\emptyset$ and A_1 containing a set of size i. Therefore

$$\ell_1! - N_1 \le (s-1)\sum_{i=1}^{\ell_1-1} \ell_1!\binom{\ell_1}{i}^{-1} = \ell_1! O\left(\frac{1}{n}\right).$$

This means that for large enough n, we have $\ell_1! \le 2N_1$. Then (6) can be continued as

$$\ell_1! N_2 \ell_2! \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right) \le 2N_1 N_2 \ell_2! \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right) = |\mathcal{Q}| \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right).$$

To summarize, we found that the contribution to the sum (5) from the subsets of A_1 and the supersets of A_{h-1} is at most

$$|\mathcal{Q}| \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right).$$

For large enough n this is smaller than $|\mathcal{Q}|\left(\Sigma(n,h) - \Sigma(n,h-1)\right)$, which means that

$$\sum_{\mathcal{C}\in\mathcal{Q}} w(\mathcal{C}) \le |\mathcal{Q}|\Sigma(n,h).$$

This completes the proof. $\square$

Remark. We had to use a weighting technique in the above proof because the usual Lubell method (proving that $\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} \le h$, and deducing $|\mathcal{F}| \le \Sigma(n,h)$ from that) does not work for this problem. To see this, let $h \ge 3$, $n \ge 2h$ and consider the following set system:

$$\mathcal{F} = \{F \in [n] \mid |F| \le h-2 \text{ or } |F| \ge n-h+2\}.$$

For $s \ge 2^{h-2}$ this set system is $Y_{h,s}$-free and $Y'_{h,s}$-free (even in the original sense, not necessarily in the rank-preserving sense). However, we have $\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} = 2(h-1) > h$.

References

[1] Boehnlein, E., and Jiang, T. Set families with a forbidden induced subposet. *Combinatorics, Probability and Computing 21*, 4 (2012), 496–511.

[2] Bukh, B. Set families with a forbidden subposet. *The Electronic Journal of Combinatorics 16*, 1 (2009), R142.

[3] De Bonis, A., Katona, G. O., and Swanepoel, K. J. Largest family without $A\cup B \subset C\cap D$. *Journal of Combinatorial Theory, Series A 111*, 2 (2005), 331–336.

[4] Erdős, P. On a lemma of Littlewood and Offord. *Bulletin of the American Mathematical Society 51*, 12 (1945), 898–902.

[5] Griggs, J. R., and Li, W.-T. The partition method for poset-free families. *Journal of Combinatorial Optimization 25*, 4 (2013), 587–596.

[6] Griggs, J. R., and Li, W.-T. Progress on poset-free families of subsets. In *Recent Trends in Combinatorics*. Springer, 2016, pp. 317–338.

[7] Griggs, J. R., and Lu, L. On families of subsets with a forbidden subposet. *Combinatorics, Probability and Computing 18*, 05 (2009), 731–748.

[8] Katona, G., and Tarján, T. G. Extremal problems with excluded subgraphs in the n-cube. In *Graph Theory*. Springer, 1983, pp. 84–93.

[9] Kleitman, D. A conjecture of Erdős-Katona on commensurable pairs among subsets of an n-set. In *Theory of Graphs, Proc. Colloq. held at Tihany, Hungary* (1966), pp. 187–207.

[10] Martin, R. R., Methuku, A., Uzzell, A., and Walker, S. A discharging method for forbidden subposet problems. *arXiv preprint arXiv:1710.05057*.

[11] Methuku, A., and Pálvölgyi, D. Forbidden hypermatrices imply general bounds on induced forbidden subposet problems. *Combinatorics, Probability and Computing* (2017), 110.

[12] Methuku, A., and Tompkins, C. Exact forbidden subposet results using chain decompositions of the cycle. *The Electronic Journal of Combinatorics 22*, 4 (2015), P4–29.

[13] Nagy, D. T. Forbidden subposet problems with size restrictions. *to appear in Journal of Combinatorial Theory, Series A*.

[14] Patkós, B. Induced and non-induced forbidden subposet problems. *The Electronic Journal of Combinatorics 22*, 1 (2015), P1–30.

[15] Sperner, E. Ein Satz über Untermengen einer endlichen Menge. *Math Z 27*, 1 (1928), 585–592.

[16] Thanh, H. T. An extremal problem with excluded subposet in the boolean lattice. *Order 15*, 1 (1998), 51–57.

[17] Tompkins, C., and Wang, Y. On an extremal problem involving a pair of forbidden posets. *manuscript*.